COHOMOLOGICAL LOCALIZATION FOR MANIFOLDS WITH BOUNDARY

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ABSTRACT. For an S^1-manifold with boundary, we prove a localization formula applying to any equivariant cohomology theory satisfying a certain algebraic condition. We show how the localization result of Kalkman and a case of the quantization commutes with reduction theorem follow easily from the localization formula.

1. INTRODUCTION

In the paper [Kal95] on cohomology rings of symplectic quotients, Kalkman uses the Cartan model of ordinary (Borel) equivariant cohomology to prove a localization result for S^1-manifolds with boundary. In this paper we extend Kalkman's result to generalized equivariant cohomology theories satisfying a certain algebraic condition. In particular, the result applies to equivariant K-theory, and in that setting it generalizes the version of "quantization commutes with reduction" proved in [Met].

We describe the basic goal briefly. Let M be a compact oriented manifold with nonempty boundary, and with a circle action that is free on ∂M. Denote the inclusion by $i : \partial M \to M$. Denote the set of fixed point components by $\mathcal{F}$. Let $X = \partial M/S^1$ and let $p : \partial M \to X$ be the quotient map. Note that this is a principal circle bundle.

Let $h_{S^1}(-)$ be a (generalized) multiplicative, complex oriented, S^1-equivariant cohomology theory, with corresponding[1] non-equivariant cohomology theory $h(-)$. Denote pullback maps in cohomology by f^* and Gysin maps (push-forwards) by $f_!$. Assume that X is h-oriented, so that the pushforward $(\pi_X)_!$ is defined, where π_X is the unique map $X \to *$. Our goal is a fixed point formula for the composed map

$$(1) \qquad \kappa : h_{S^1}(M) \xrightarrow{i^*} h_{S^1}(\partial M) \xrightarrow{(p^*)^{-1}} h(X) \xrightarrow{(\pi_X)_!} h(*).$$

This is given in Theorem 1.

In Section 2 we present a simple geometric construction that converts the boundary into a fixed point component. In Section 3 we apply the usual localization theorem to the new manifold. In Section 4 we introduce the idea of a residue map, which requires an additional assumption on the coefficient ring of the equivariant cohomology theory in question, and use the residue in Section 5 to get the desired formula. In Section 6 we first show how this formula specializes to Kalkman's in the case of Borel cohomology. We then show that in K-theory it leads to a proof of a restricted case of the "quantization commutes with reduction" result in symplectic geometry.

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[1]See section 3.

In a forthcoming paper we will investigate Theorem 1 in other cohomology theories such as cobordism and elliptic cohomology. We will also give a similar theorem for actions of $SU(2)$.

2. A Geometric Construction

Given an S^1-manifold M, with notation as in the introduction, we use a simple procedure analogous to blowing up (and to symplectic cutting [Ler95]) to produce a new manifold without boundary which adds X to the fixed point set.

Note: throughout the paper, by S^1 we really mean $S^1 \subset \mathbb{C}$, with the attendant explicit identification of $\mathrm{Lie}(S^1)$ with $\mathbb{R}$ ($\exp(t) = e^{it}$, $t \in \mathbb{R}$). This is important for orientation issues.

Since the principal stratum for a compact group action is open and dense, there is a neighborhood U of ∂M on which the action of S^1 is free. In fact, we can assume that U is invariant and that there is an equivariant homeomorphism $\phi :$ $U \cong \partial M \times [0, \infty)$. Consider the action of S^1 on $\mathbb{C}$ of weight -1 (given $\lambda \in S^1 \subset \mathbb{C}$, $\lambda \cdot z = \lambda^{-1} z$) and form the associated bundle

$$E = \partial M \times_{S^1} \mathbb{C}.$$

(Here we divide by the diagonal action of S^1.) This is an S^1-equivariant vector bundle, with S^1-action defined by $\lambda \cdot [m, z] = [m, \lambda z]$. Now ϕ induces a map

$$(2) \qquad\qquad \psi : U \setminus \partial M \to E \setminus X \cong \partial M \times_{S^1} \mathbb{C}^*$$

$$(3) \qquad\qquad m \mapsto [\phi_1(m), \phi_2(m)]$$

which is easily seen to be a homeomorphism (since $\mathbb{C}^*/S^1 \cong (0, \infty)$). It is also S^1-equivariant, since

$$\begin{aligned}
\psi(\lambda \cdot m) &= [\phi_1(\lambda \cdot m), \phi_2(\lambda \cdot m)] \\
&= [\lambda \cdot \phi_1(m), \phi_2(m)] \\
&= [\phi_1(m), \lambda \phi_2(m)]
\end{aligned}$$

where the last equality comes from our taking the weight -1 action to form the bundle E.

We use the map ψ to glue, obtaining

$$\hat{M} = (M \setminus \partial M) \cup_\psi E.$$

Clearly $\hat{M}$ is a compact boundaryless S^1-manifold, with fixed point set equal to the union of the fixed point set of M and $\partial M/S^1 = X$. $\hat{M}$ is oriented, but we have to be slightly careful about that. The orientation on M gives an orientation on ∂M in the usual way. Since S^1 is oriented, we get an orientation on the quotient X. We also have an orientation on the complex vector bundle E over X, but we need to use the opposite orientation from the standard complex orientation, in order to make ψ orientation-preserving Hence when we glue, we get an orientation on $\hat{M}$. Note that the normal bundle of X in $\hat{M}$ is exactly E, oriented by the conjugate complex orientation.

In the simplest example, M is the upper hemisphere of S^2 with the standard rotational action of S^1. Then $\hat{M}$ is clearly S^2 with the analogous action. We note that, given any orientation of S^2, the circle action rotates in the positive direction (with respect to the orientation) at one pole, and in the negative direction at the other. This is an example of the orientation issue discussed above.

The above description is the best for seeing the geometry of $\hat{M}$ (in particular that it is a smooth manifold), but an alternate description shows that it is a very natural topological construction as well.

Proposition 1. $\hat{M}$ *is homeomorphic to the homotopy pushout of the diagram*

$$
\begin{array}{ccc}
\partial M & \overset{i}{\hookrightarrow} & M \\
\downarrow{\scriptstyle p} & & \\
X. & &
\end{array}
$$

Proof. To take the homotopy pushout we must replace p by its mapping cylinder

$$\partial M \overset{j}{\hookrightarrow} \frac{(\partial M \times I) \cup X}{(m,1) \sim p(m)} = \mathrm{Cyl}(p) \cong D(E)$$

where the last term denotes the disc bundle of E. Hence when we take the ordinary pushout

$$\frac{M \cup \mathrm{Cyl}(p)}{i(m) \sim j(m)}$$

we are simply gluing M to the disc bundle of E, which gives a space homeomorphic to $\hat{M}$. $\qquad\square$

3. LOCALIZATION ON $\hat{M}$

Let $h_{S^1}(-)$ be an S^1-equivariant, complex oriented multiplicative cohomology theory, free and split over the cohomology theory $h(-)$, as in [May96]. This implies the following facts: the equivariant coefficient ring $h_{S^1} = h_{S^1}(*)$ is an algebra over $h = h(*)$. Second, for a trivial S^1-space X, we have a natural isomorphism

$$(4) \qquad\qquad h_{S^1}(X) \cong h(X) \otimes_h h_{S^1}.$$

Third, for a free S^1-space Y, the projection map p induces an isomorphism $h(Y/S^1) \cong h_{S^1}(Y)$; more precisely the composed map

$$h(Y/S^1) \to h(Y/S^1) \otimes h_{S^1} \cong h_{S^1}(Y/S^1) \to h_{S^1}(Y)$$

(the first map is just $\alpha \mapsto \alpha \otimes 1$), which we will denote by p^*, is an isomorphism. Examples are Borel cohomology [Bor60], equivariant K-theory [Seg68], and (homotopical) equivariant bordism [May96], [Sin].

We note a simple relation between $h_{S^1}(M)$ and $h_{S^1}(\hat{M})$.

Proposition 2. *For every class $\alpha \in h_{S^1}(M)$ there is a unique class $\hat{\alpha} \in h_{S^1}(\hat{M})$ such that*

$$(5) \qquad\qquad \hat{\alpha}|(M \setminus \partial M) = \alpha|(M \setminus \partial M)$$

$$(6) \qquad\qquad \hat{\alpha}|X = (p^*)^{-1}i^*\alpha.$$

Proof. Observe that the Mayer-Vietoris sequence for $\hat{M}$ which we get from Prop. 1 is

$$(7) \qquad \cdots \overset{\partial}{\longrightarrow} h_{S^1}^q(\hat{M}) \longrightarrow h_{S^1}^q(M) \oplus h_{S^1}^q(X) \longrightarrow h_{S^1}^q(\partial M) \overset{\partial}{\longrightarrow} \cdots .$$

However, since $h_{S^1}(X) \cong h(X) \otimes h_{S^1}$, and the map $p^* : h(X) \to h_{S^1}(\partial M)$ is an isomorphism, the map

$$h_{S^1}^q(M) \oplus h_{S^1}^q(X) \to h_{S^1}^q(\partial M)$$

in the Mayer-Vietoris sequence is surjective in every degree, forcing the coboundary map to vanish. Hence the long exact sequence splits into short exact sequences

$$(8) \qquad 0 \longrightarrow h_{S^1}^q(\hat{M}) \longrightarrow h_{S^1}^q(M) \oplus h_{S^1}^q(X) \longrightarrow h_{S^1}^q(\partial M) \longrightarrow 0.$$

Given $\alpha \in h_{S^1}(M)$, let $\beta = (p^*)^{-1} i^* \alpha \in h(X)$. Clearly the pair $(\alpha, \beta \otimes 1)$ maps to zero in the Mayer-Vietoris sequence, and so defines a unique element $\hat{\alpha} \in h_{S^1}(\hat{M})$, satisfying (5). ∎

The class $\hat{\alpha}$ "glues" α to its "quotient" β.

We now need to deal with h_{S^1}-orientations. In this case we can define a reasonable notion of an h_{S^1}-orientation on the manifold-with-boundary M. First recall that $\hat{M}^{S^1} = M^{S^1} \cup X$, and that the normal bundle of X in $\hat{M}$ is exactly E. We use its conjugate complex structure $\bar{E}$ to determine its h_{S^1}-orientation, to be consistent with the ordinary orientation, as discussed in Section 2. This leads to the

Definition 1. *Let h_{S^1} be an S^1-equivariant cohomology theory as above. Let M be a manifold with boundary, with S^1-action free on ∂M, and construct $\hat{M}$ as above. Assume that $X = \partial M/S^1$ has a given h-orientation and that each $F \in \mathcal{F}$ and each normal bundle ν_F have given h_{S^1}-orientations.*

*Then an h_{S^1}-**orientation** on M compatible with this data is defined to be an orientation on $\hat{M}$, compatible with the orientations on F and ν_F in the usual way, and compatible with the orientation on $X \subset \hat{M}$ and the conjugate complex orientation on $E \cong \nu_X$.*

In the case of ordinary Borel cohomology, an h_{S^1}-orientation on a manifold without boundary is simply an ordinary (geometric) orientation. From the discussion in Section 2 it is clear that a geometric orientation on M gives a Borel h_{S^1}-orientation as defined above.

We seek a fixed point formula for the map

$$(9) \qquad\qquad h_{S^1}(M) \xrightarrow{i^*} h_{S^1}(\partial M) \xrightarrow{(p^*)^{-1}} h(X) \xrightarrow{(\pi_X)_!} h(*).$$

Any multiplicative equivariant cohomology theory has a localization theorem [tD71], [Kaw91], which gives a fixed point formula for

$$h_{S^1}(\hat{M}) \xrightarrow{(\pi_{\hat{M}})_!} h_{S^1}.$$

Specifically, it says that after sufficiently localizing (algebraically) in the ring h_{S^1} and the h_{S^1}-module $h_{S^1}(\hat{M})$, we have

$$(10) \qquad\qquad (\pi_{\hat{M}})_! \hat{\alpha} = \sum_{F \in \hat{\mathcal{F}}} (\pi_F)_! \frac{i_F^* \hat{\alpha}}{\tilde{e}(\nu_F)}$$

where $\hat{\mathcal{F}}$ is the set of fixed point components of $\hat{M}$, $i_F : F \to M$ is the inclusion map and $\tilde{e}$ is the h_{S^1} Euler class (the restriction of the h_{S^1} Thom class to the zero section). The algebraic localization necessary inverts the following subset S of h_{S^1}. Let $\mathcal{R}$ be the set of finite dimensional, h_{S^1}-oriented representations of S^1 with no trivial summand (where a representation V is regarded as an equivariant bundle over a point). Let

$$S = \{\tilde{e}(V) \mid V \in \mathcal{R}\}.$$

Then equation (10) holds in the ring $S^{-1} h_{S^1}$.

The h_{S^1}-orientation on M, and the resulting orientations on $\hat{M}$, X, F, and ν_F allow us to use the localization formula. Note that the S^1-action on $\bar{E}$ has weight -1. We saw above that $\hat{\alpha}|_X = (p^*)^{-1}i^*\alpha$, hence

$$(11) \qquad (\pi_{\hat{M}})_!\hat{\alpha} = \sum_{F \in \mathcal{F}} (\pi_F)_! \frac{i_F^*\alpha}{\tilde{e}(\nu_F)} + (\pi_X)_! \frac{(p^*)^{-1}i^*\alpha}{\tilde{e}(\bar{E})}.$$

This is our key formula, but note the presence of the unknown quantity $(\pi_{\hat{M}})_!\hat{\alpha}$. The next step is to get rid of it, and get rid of the factor $(\tilde{e}(\bar{E}))^{-1}$. To do that we need a further assumption about h_{S^1}, which goes beyond the usual axioms for equivariant cohomology theories, but is satisfied, for example, by Borel cohomology and equivariant K-theory.

4. RESIDUE MAPS

First we need a notation for the universal Euler classes of the theory. Let $\mathcal{L} \to \mathbb{CP}^N$ ($N \in \mathbb{N}$) be the universal line bundle, where both $\mathcal{L}$ and $\mathbb{CP}^N$ have the trivial S^1-action. Let $\mathbb{C}_m$ be the weight m irreducible representation of S^1, given by $\lambda \cdot z = \lambda^m z$. Let $\mathcal{L}_m = \mathcal{L} \otimes \mathbb{C}_m$. Denote the h_{S^1}-Euler class of $\mathcal{L}_m$ by

$$E_m = \tilde{e}(\mathcal{L}_m) \in h_{S^1}(\mathbb{CP}^N) \cong h_{S^1}[y]/(y^{N+1}).$$

Here $y = e(L)$. (We suppress N from the notation because it will be irrelevant to what follows, as long as it is large enough. However we do not use $\mathbb{CP}^\infty$ because we need y to be nilpotent.)

Denote the canonical localization map by $\varepsilon : h_{S^1} \to S^{-1}h_{S^1}$. We will need the existence of the following kind of map.

Definition 2. *A **residue map** for $h_{S^1}(-)$ is an h-module homomorphism $\rho : S^{-1}h_{S^1} \to h$ such that:*

1. *$\rho \circ \varepsilon = 0$.*
2. *The natural extension of ρ to a map $\rho : S^{-1}h_{S^1}[y]/(y^{N+1}) \to h[y]/(y^{N+1})$ satisfies*

$$\rho((E_{-1})^{-1}) = -1.$$

A simple naturality argument shows that condition 2 implies the following: given a trivial S^1-space X and L an S^1-equivariant line bundle of weight -1 over X, the map $1 \otimes \rho$ takes $(\tilde{e}(L))^{-1} \in h(X) \otimes S^{-1}h_{S^1}$ to $-1 \in h(X) \otimes h \cong h(X)$. This is how we will use condition 2.

We now give the conditions under which such a residue map exists, in terms of the the formal group law of h_{S^1}. Let F be the formal group law of h_{S^1}, i.e. $F(X,Y) \in h_{S^1}[[X,Y]]$ is the formal power series such that for two line bundles L_1, L_2 on X,

$$(12) \qquad \tilde{e}(L_1 \otimes L_2) = F(\tilde{e}(L_1) \otimes \tilde{e}(L_2)).$$

It is well-known that

$$F(X,Y) = \sum_{k,l=0}^{\infty} a_{k,l}X^kY^l = X + Y + XY\hat{F}(X,Y)$$

where $\hat{F}(X,Y) \in h_{S^1}[[X,Y]]$ and $\hat{F}(X,Y) = \hat{F}(Y,X)$. Hence $a_{k0} = a_{0k} = \delta_{1k}$. Let $e = \tilde{e}(\mathbb{C}_{-1})$. Then

$$(13) \qquad E_{-1} = \tilde{e}(\mathbb{C}_{-1} \otimes L) = F(e,y)$$

$$(14) \qquad = \sum_{k,l=0}^{\infty} a_{kl}e^k y^l$$

$$(15) \qquad = \sum_{l=0}^{\infty} b_l y^l$$

where $b_l \in h_{S^1}$ are defined by the last equation, or by directly calculating $\tilde{e}(\mathbb{C}_{-1} \otimes L) \in h_{S^1}(\mathbb{CP}^N)$. Since $a_{k0} = a_{0k} = \delta_{1k}$, $b_0 = e$. Hence

$$(16) \qquad (E_{-1})^{-1} = e^{-1}[1 + \sum_{l=1}^{\infty}(b_l/e)y^l]^{-1}$$

$$(17) \qquad = e^{-1} \sum_{k=0}^{\infty} \left(-\sum_{l=1}^{\infty}(b_l/e)y^l \right)^k$$

$$(18) \qquad = e^{-1} + \sum_{k=1}^{\infty} -e^{-1} \left(\sum_{l=1}^{\infty}(b_l/e)y^l \right)^k$$

$$(19) \qquad = e^{-1} + \sum_{k=1}^{\infty} c_k y^k$$

where the coefficients $c_k \in h_{S^1}[e^{-1}]$ are determined by the last equation. The sums are actually finite for any given N.

Let M be the h-submodule of $S^{-1}h_{S^1}$ generated by the c_k, let

$$N = \frac{S^{-1}h_{S^1}}{h_{S^1} + M},$$

and let A be the h-submodule of N generated by e^{-1}.

Proposition 3. *A residue map ρ exists for $h_{S^1}(-)$ if and only if e^{-1} is not torsion in N as an h-module and the inclusion $A \subset N$ splits as a map of h-modules.*

If a residue map exists then the set of possible residue maps is given by $\mathrm{Hom}_h(N/A, h)$.

Proof. In the definition of a residue, clearly condition (1) is satisfied iff ρ defines a map from $S^{-1}h_{S^1}/h_{S^1}$ to h. Since

$$\rho((E_{-1})^{-1}) = \rho(e^{-1}) + \sum_{k=1}^{\infty} \rho(c_k)y^k,$$

condition (2) will be satisfied iff $\rho(c_k) = 0$ for all $k \geq 1$ and $\rho(e^{-1}) = -1$. The former says that ρ defines a map from $N = S^{-1}h_{S^1}/(h_{S^1} + M)$ to h, while the latter requires that this map split the inclusion of $A = \langle e^{-1} \rangle$ into N. The set of such splittings is just $\mathrm{Hom}_h(N/A, h)$. $\square$

In particular, when h is a field, the conditions for existence are automatically satisfied as long as $e^{-1} \notin h_{S^1} + M$.

We now show that our two main examples have such a map ρ. We will consistently work over $\mathbb{C}$, without further comment.

4.1. **Borel.** Borel cohomology is defined by

$$H_{S^1}(M) = H(M \times_{S^1} ES^1, \mathbb{C}).$$

This is what is usually referred to in the geometry literature simply as "equivariant cohomology." It can be computed by the deRham-theoretic Cartan model (see [AB84]). We have $h_{S^1} \cong \mathbb{C}[u]$, $e = -u$, $S = \{(mu)^k \mid m \neq 0, k \geq 0\}$. Hence $S^{-1}h_{S^1} \cong \mathbb{C}[u, u^{-1}]$. The formal group law is the additive one, $F(X, Y) = X + Y$, so $\hat{F} = 0$. Therefore

$$(20) \qquad (E_{-1})^{-1} = (-u + y)^{-1}$$

$$(21) \qquad = -u^{-1}(1 - (y/u))^{-1}$$

$$(22) \qquad = -u^{-1} + \sum_{k=1}^{\infty} y^k u^{-k-1}$$

so $c_k = u^{-k-1}$, $k \geq 1$, and $M = \langle u^{-k-1} \mid k \geq 1 \rangle$. We have

$$h_T + M = \langle u^l \mid l \in \mathbb{Z}, l \neq -1 \rangle$$

so $N \cong \langle u^{-1} \rangle$ and the inclusion $A \subset N$ is in fact an isomorphism. Hence there is a unique residue map

$$\rho\left(\sum_{k=-M}^{M'} a_k u^k\right) = a_{-1},$$

or in complex-analytic terms, $\rho(f) = \operatorname{Res}_{u=0}(f(u)\, du)$.

4.2. *K*-**theory.** Here $h \cong \mathbb{C}$ and the equivariant coefficient ring is $R(S^1) \otimes \mathbb{C} \cong \mathbb{C}[u, u^{-1}] \cong \mathbb{C}[e, (1 - e)^{-1}]$ where $e = \tilde{e}(\mathbb{C}_{-1}) = 1 - u^{-1}$. (See [AS68], [Seg68].) S is the multiplicative subset generated by $\{1 - u^k \mid k \neq 0\}$, so

$$S^{-1}h_{S^1} \cong \mathbb{C}[u, u^{-1}, \{(u - \gamma)^{-1} \mid \gamma \text{ a root of unity }\}].$$

The formal group law is $F(X, Y) = X + Y - XY$, so $\hat{F} = -1$, or $b_0 = e$, $b_1 = 1 - e$. Therefore

$$(E_{-1})^{-1} = e^{-1} + \sum_{k=1}^{\infty} (-y)^k e^{-1}(e^{-1} - 1)^k$$

and

$$(23) \qquad c_k = e^{-1}(1 - e^{-1})^k = \frac{1}{1 - u^{-1}}\left(1 - \frac{1}{1 - u^{-1}}\right)^k$$

$$(24) \qquad = \frac{-u}{(1 - u)^{k+1}}$$

$$(25) \qquad = \frac{1}{(1 - u)^k} - \frac{1}{(1 - u)^{k+1}}.$$

Let B be the vector subspace generated by $\{(1 - u)^{-k} \mid k \geq 1\}$ and let D be generated by

$$\{(u - \gamma)^{-k} \mid k \geq 1, \gamma \text{ a root of unity}, \gamma \neq 1\}.$$

Let $\sigma : B \to \mathbb{C}$ be defined by

$$\sigma\left(\sum a_k(1 - u)^{-k}\right) = \sum a_k.$$

Then $S^{-1}h_{S^1} \cong h_{S^1} \oplus B \oplus D$, and $M = \operatorname{Ker}\sigma$. Hence

$$N = \frac{S^{-1}h_{S^1}}{h_{S^1} + M} \cong B/M \oplus D \cong \mathbb{C} \oplus D$$

where the second isomorphism is induced by σ. Under this map

$$e^{-1} = \frac{1}{1 - u^{-1}} = 1 - \frac{1}{1 - u} \mapsto -1 \in \mathbb{C} \subset \mathbb{C} \oplus D.$$

Hence any extension of σ to $B \oplus D$ is a residue map. The set of possible residue maps is naturally isomorphic to D^*.

There are two simple choices for the map ρ.

Lemma 1. *The map $\rho_1 : \mathbb{C}(u) \to \mathbb{C}$ defined by $\rho(f) = -\operatorname{Res}_{u=1}(u^{-1}f(u)\,du)$ restricts to $S^{-1}h_{S^1}$ to be the unique residue map vanishing on D.*

Proof. The map ρ_1 evidently vanishes on $h_{S^1} \cong \mathbb{C}[u, u^{-1}]$ and D since every element of either set has no pole at $u = 1$. On B,

$$\frac{u^{-1}}{(1-u)^k} = (1-u)^{-k}[1 - (1-u)]^{-1}$$

$$= (1-u)^{-k}\sum_{l=0}^{\infty}(1-u)^l$$

$$= \sum_{l=0}^{\infty}(-1)^{l-k}(u-1)^{l-k}$$

so

$$\rho_1((1-u)^{-k}) = -\operatorname{Res}_{u=1}(u^{-1}(1-u)^{-k}\,du) = -(-1)^{-1} = 1 = \sigma((1-u)^{-k}).$$

So ρ_1 extends σ as desired, hence is the required residue map.   □

There is a more convenient residue map for the application to quantization in Section 6. Let $L_+ = \mathbb{C}[[u]][u^{-1}]$ and $L_- = \mathbb{C}[[u^{-1}]][u]$ be the rings of positive and negative Laurent series. Given a rational function $f \in \mathbb{C}(u)$, define $L_{+,0}(f)$ to be the constant term in the positive Laurent series for f, and define $L_{-,0}(f)$ to be the constant term in the negative Laurent series for f. Define $\rho_{0,\infty} : \mathbb{C}(u) \to \mathbb{C}$ by

$$\rho_{0,\infty} = L_{+,0} - L_{-,0}.$$

In complex analytic terms,

$$\rho_{0,\infty}(f) = (\operatorname{Res}_{u=0} + \operatorname{Res}_{u=\infty})(u^{-1}f(u)\,du)$$

Lemma 2. *The map $\rho_{0,\infty}$ restricts to $S^{-1}h_{S^1}$ to be a residue map, distinct from ρ_1.*

Proof. The residue theorem implies that for a function f with poles only at $0, 1, \infty$, $\rho_1(f) = \rho_{0,\infty}(f)$. Hence the two maps agree on $S^{-1}h_{S^1}/D$, so $\rho_{0,\infty}$ is a residue map. However it is easy to see that they disagree on D. In fact a simple calculation shows that

$$\rho_{0,\infty}((u-\gamma)^{-k}) = (-\gamma)^{-k} \neq 0.$$

□

It is interesting that in both these cases, the equivariant coefficient rings are integral domains, and the residue maps extend easily to the entire field of fractions $\mathbb{C}(u)$.

5. The Localization Formula

Theorem 1. *Let $h_{S^1}(-)$ be an S^1-equivariant cohomology theory, free and split over $h(-)$, and suppose that ρ is a residue map for $h_{S^1}(-)$. Let M be an S^1-manifold with boundary ∂M on which S^1 acts freely, and let $X = \partial M/S^1$. Assume that M is h_{S^1}-oriented, and define κ as in the introduction. Then for every $\alpha \in h_{S^1}(M)$,*

$$(26) \qquad \kappa(\alpha) = \sum_{F \in \mathcal{F}} (\pi_F)_! \rho \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right).$$

Proof. Recall that ordinary localization on $\hat{M}$ gave (11):

$$(\pi_{\hat{M}})_! \hat{\alpha} = \sum_{F \in \mathcal{F}} (\pi_F)_! \frac{i_F^* \alpha}{\tilde{e}(\nu_F)} + (\pi_X)_! \frac{(p^*)^{-1} i^* \alpha}{\tilde{e}(\bar{E})}.$$

Now $(\pi_{\hat{M}})_! \hat{\alpha} \in h_{S^1}$, so applying ρ to (11) gives (writing ρ in place of $1 \otimes \rho$)

$$(27) \qquad 0 = \sum_{F \in \mathcal{F}} \rho(\pi_F)_! \frac{i_F^* \alpha}{\tilde{e}(\nu_F)} + \rho(\pi_X)_! \frac{(p^*)^{-1} i^* \alpha}{\tilde{e}(E)}$$

$$(28) \qquad = \sum_{F \in \mathcal{F}} (\pi_F)_! \rho \frac{i_F^* \alpha}{\tilde{e}(\nu_F)} - (\pi_X)_! (p^*)^{-1} i^* \alpha$$

because ρ is an h-module map. The last term is just $\kappa(\alpha)$, so the theorem is proved. $\square$

6. First Applications

First we show how this result agrees with and extends known results in the Borel and K-theory cases.

6.1. **Kalkman's Result.** In this case the result is exactly Kalkman's formula, which he proves using the Cartan Model. We compare it to the form given in [GK96].

We have already remarked on the fact that an ordinary geometric orientation of M gives exactly the kind of orientation we need in the Borel theory.

Their map r (the Kirwan map) is defined to be $(p^*)^{-1} \circ i^*$. Their map "res" is exactly our ρ for the Borel case. Hence

$$(29) \qquad \int_X r(\alpha) = \kappa(\alpha)$$

$$(30) \qquad = \sum_{F \in \mathcal{F}} (\pi_F)_! \rho \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right)$$

$$(31) \qquad = \sum_{F \in \mathcal{F}} \int_F \text{res} \left(\frac{\alpha|_F}{\tilde{e}(\nu_F)} \right)$$

which is exactly Kalkman's localization formula.

6.2. **Quantization Commutes With Reduction.** The most interesting application of the formula in K-theory is in the case of a Hamiltonian circle action. Let (M, ω) be a symplectic manifold with a Hamiltonian circle action, with moment map $\phi : M \to \mathrm{Lie}(S^1)^* \cong \mathbb{R}$. Assume that the symplectic form is integral. Then it is well-known that there is a "prequantum" line bundle L over M, with an action of S^1 lifting the given one on M, whose Chern class is $[\omega]$. Further, the action of S^1 on L over a fixed point component is simple to describe (see [Kos70]): on a component F, the moment map takes a fixed, integral value $\phi(F)$. The weight of the action of S^1 on the fibers of the line bundle is exactly $-\phi(F)$.

Any symplectic manifold has a natural isotopy class of complex structures associated to it ([MS95]). Choose such a structure. Since the circle action is symplectic, it preserves the complex structure. We will actually use the conjugate of this complex structure as the K_{S^1}-orientation on M. (This is to agree with an index-theoretic definition of the quantization—see [Met] for details.) The *quantization* of the space M is defined to be the pushforward $Q(M) := (\pi_M)_!([L]) \in R(S^1)$.

When we have a Hamiltonian group action, it is natural to perform symplectic reduction. The reduced space is defined to be

$$M_0 = \phi^{-1}(0)/S^1$$

which is a smooth symplectic manifold if 0 is a regular value of ϕ and the action of S^1 on the zero level set is free; we will assume these conditions hold for the remainder of the section. If M is prequantizable then so is M_0, where the prequantum line bundle on the reduced space is obtained by restricting L to the zero level set $\phi^{-1}(0)$, and then identify this with a (non-equivariant) line bundle L_0 downstairs.

Since M_0 is symplectic, it has a complex structure, and we again use the conjugate structure to give it a K-orientation. The quantization of M_0 is the pushforward $(\pi_{M_0})_!(L_0) \in \mathbb{Z}$.

Theorem 2 ([DGMW95]). *Given a prequantizable Hamiltonian S^1 space with a free action on the zero level set, we have*

$$Q(M_0) = Q(M)_0,$$

where the right hand side stands for the multiplicity of the trivial representation in the virtual representation $Q(M)$.

We will derive this theorem from a more general result, Theorem 3 below, which is an extension of a result originally due to A. Canas da Silva, Y. Karshon and S. Tolman [CKT]. They consider three kinds of quantization of presymplectic manifolds, including the case of $\mathrm{Spin}_{\mathbb{C}}$-quantization. We will phrase our generalization of their main result in K-theoretic terms.

The data for this version of quantization are: a compact S^1-manifold M, a fixed K_{S^1}-orientation on M, and a class $\alpha \in K_{S^1}(M)$. The quantization of M is then defined to be the pushforward

$$Q(M) = (\pi_M)_!\alpha \in K_{S^1}(*) \cong R(S^1).$$

To define a general version of reduction, we follow [CKT] (with minor modifications in notation) and make the

Definition 3. *A **reducible hypersurface** in M is a co-oriented submanifold Z of codimension one which is invariant under the S^1-action and on which this action is free. The **reduction** of M at Z is the quotient $M_{\mathrm{red}} = Z/S^1$. Z is **splitting** if M*

*is the union of two (not necessarily connected) manifolds with boundary, say M_+,
M_-, along their common boundary Z, such that positive (negative) normal vectors
to Z point into M_+ (M_-). We then say that Z **splits** M into M_+, M_-.*

For example, if $\phi : M \to \mathbb{R}$ is an invariant map, with 0 a regular value, and
if S^1 acts freely on $Z = \phi^{-1}(0)$, then Z splits M into $M_+ = \phi^{-1}([0, \infty))$ and
$M_- = \phi^{-1}((-\infty, 0])$. The main example we have in mind is where M is Hamiltonian
and ϕ is a moment map.

Let Z be a splitting, reducible hypersurface in M. We have exact sequences

$$(32) \qquad 0 \longrightarrow TZ \longrightarrow i^*TM \longrightarrow \nu_Z \longrightarrow 0$$

$$0 \longrightarrow V \longrightarrow TZ \longrightarrow \pi^*(TM_{\mathrm{red}}) \longrightarrow 0$$

where V, the vertical subbundle, is the same as the set of vectors that are tangent
to the orbits of S^1. Note that V and ν_Z are both oriented trivial real line bundles.
Hence

$$i^*TM \cong \pi^*TM_{\mathrm{red}} \oplus \mathbb{R}^2.$$

The K_{S^1}-orientation on M hence gives a K_{S^1}-orientation on π^*TM_{red}, hence a
K-orientation on M_{red}.

The class α descends to a class $\alpha_{\mathrm{red}} \in K(M_{\mathrm{red}})$, first by restriction to Z, and
then by the isomorphism $K_{S^1}(Z) \cong K(M_{\mathrm{red}})$. Hence we can define the quantization
of M_{red} exactly as above,

$$Q(M_{\mathrm{red}}) = (\pi_{M_{\mathrm{red}}})_! \alpha \in \mathbb{Z}.$$

This defines the quantization of the reduction of M; the reduction of the quantiza-
tion of M is defined to be $Q(M)_0$, the multiplicity of the trivial representation in
$Q(M)$. Theorem 3 will say that these are equal, subject to certain conditions on α
and the normal bundles of the fixed point components of M.

In order to get the sharpest version of the theorem, we will be a little tricky about
orientations on the fixed point components. Given a component $F \subset M^{S^1}$ with
$F \subset M_+$, equip the normal bundle ν_F with a complex structure by declaring all of
the weights of the S^1-action to be positive, and use the associated K_{S^1}-orientation.
Together with the given K_{S^1}-orientation on M, this gives an K_{S^1}-orientation on F.
For $F \subset M_-$, use the complex structure and associated K_{S^1}-orientation induced
by declaring the weights to be negative.

It is not hard to see that the K_{S^1}-orientation on M gives a K_{S^1}-orientation on
M_+, M_-, in the sense of Definition 1, compatible with the above orientations for
the fixed point components and the reduced space.

Given a connected component F of M^{S^1}, let

$$w(F) = \sum |\text{weights of } S^1\text{-action on } \nu_F|.$$

Note that the absolute value of a weight is well-defined independent of the complex
structure chosen.

Recall that

$$K_{S^1}(F) \cong K(F) \otimes R(S^1) \cong K(F) \otimes \mathbb{Z}[u, u^{-1}].$$

Hence for $\alpha \in K_{S^1}(M)$,

$$i_F^* \alpha = \sum_{\text{Min}(\alpha,F)}^{\text{Max}(\alpha,F)} \alpha_{F,k} u^n$$

for some integers $\text{Min}(\alpha, F), \text{Max}(\alpha, F)$. (Note that if α_F is homogeneous in u, then $\text{Min}(\alpha, F) = \text{Max}(\alpha, F) = \deg(i_F^* \alpha)$.)

We can now state the

Theorem 3. *Let M be a smooth, compact, K_{S^1}-oriented S^1-manifold and let $\alpha \in K_{S^1}(M)$. Let Z be a reducible hypersurface which splits M into M_+ and M_-. Assume that the following conditions hold on the fixed point components $F \subset M^{S^1}$:*

(33)   *If $F \subset M_+$, $\text{Max}(\alpha, F) < w(F)$;*

(34)   *If $F \subset M_-$, $\text{Min}(\alpha, F) > -w(F)$.*

Then

$$Q(M)_0 = Q(M_{\text{red}}).$$

Proof. We will apply Theorem 1 to the manifold with boundary M_+, whose boundary is Z, and apply the ordinary localization theorem to all of M. Clearly

$$Q(M_{\text{red}}) = \kappa(\alpha|_{M_+})$$

so Theorem 1 gives

$$(35) \qquad Q(M_{\text{red}}) = \sum_{F \subset M_+} (\pi_F)_! \rho \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right)$$

$$(36) \qquad = \sum_{F \subset M_+} (\pi_F)_! (L_{+,0} - L_{-,0}) \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right)$$

while

$$(37) \qquad Q(M)_0 = ((\pi_M)_! \alpha)_0 = \left[\sum_{F \subset M} (\pi_F)_! \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) \right]_0$$

$$(38) \qquad = \sum_{F \subset M} (\pi_F)_! (L_{+,0}) \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right)$$

so

(39)

$$Q(M_{\text{red}}) - Q(M)_0 = -\sum_{F \subset M_+} (\pi_F)_! L_{-,0} \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) - \sum_{F \subset M_-} (\pi_F)_! L_{+,0} \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right).$$

We will analyze the first term; the second is similar. Recall that by our conventions, the weights of ν_F for $F \subset M_+$ are positive. Hence, letting $\nu_{F,k}$ be the weight k

isotypic part of ν_F, the K_{S^1}-Euler class is

$$\tilde{e}(\nu_F) = \tilde{e}(\sum_{k>0} \nu_{F,k} u^k)$$

$$= \prod_{k>0} \tilde{e}(\nu_{F,k} u^k)$$

$$= \prod_{k>0} \sum_{l=0}^{\mathrm{rk}(\nu_{F,k})} (-1)^l (\Lambda^l \nu_{F,k}) u^{kl}.$$

Each factor is a polynomial in u with constant term 1, invertible leading coefficient, and degree $k \cdot \mathrm{rk}(\nu_{F,k})$. The sum of these degrees is $w(F)$.

Lemma 3. *Let R be a ring. Let $P \in R[u]$ have invertible leading coefficient and constant coefficient and degree n. Then P is invertible in the Laurent ring $R[[u^{-1}]][u]$, and moreover, $L_-(P^{-1}) \in R[[u^{-1}]][u]$ actually lies in $u^{-n}R[[u^{-1}]]$.*

Proof of the Lemma. When $R = \mathbb{C}$, this is clear from examining the poles and order of vanishing of P^{-1}. The same holds for any ring R by direct calculation:

$$P = P_n u^n + P_{n-1} u^{n-1} + \ldots + P_0$$

$$P = P_n u^n (1 + (P_{n-1}/P_n) u^{-1} + \ldots + (P_0/P_n) u^{-n})$$

$$P^{-1} = P_n^{-1} u^{-n} \sum_{l=0}^{\infty} (-1)^l ((P_{n-1}/P_n) u^{-1} + \ldots + (P_0/P_n) u^{-n})^l \in u^{-n} R[[u^{-1}]].$$

$\square$

Hence

$$L_-(\tilde{e}(\nu_{F,k} u^k))^{-1} \in u^{k \cdot \mathrm{rk}(\nu_{F,k})} K(F)[[u]]$$

and

$$L_-(\tilde{e}(\nu_F))^{-1} \in u^{w(F)} K(F)[[u]].$$

Therefore, for $F \subset M_+$,

$$L_- \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) \in u^{-w(F) + \mathrm{Max}(\alpha, F)} K(F)[[u^{-1}]].$$

Given the condition (33),

$$L_- \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) \in u^{-1} K(F)[[u^{-1}]]$$

so

$$L_{-,0} \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) = 0.$$

Similarly, condition (34) implies that for $F \subset M_-$,

$$L_{+,0} \left(\frac{i_F^* \alpha}{\tilde{e}(\nu_F)} \right) = 0.$$

Hence

$$Q(M_{\mathrm{red}}) - Q(M)_0 = 0.$$

$\square$

This easily implies Theorem 2.

Proof of Theorem 2. $Z = \phi^{-1}(0)$ splits M into $M_+ = \phi^{-1}((0, \infty))$, $M_- = \phi^{-1}((-\infty, 0))$. For any fixed point component F, we have

$$i_F^* \alpha = \alpha_F u^{-\phi(F)}$$

so $\mathrm{Max}(\alpha, F) = \mathrm{Min}(\alpha, F) = -\phi(F)$ which is negative on M_+, positive on M_-. Hence both conditions in Theorem 3 are satisfied, and

$$Q(M_{\mathrm{red}}) = Q(M)_0.$$

$\square$

References

[AB84]   M. F. Atiyah and Raoul Bott. The moment map and equivariant cohomology. *Topology*, 23:1–28, 1984.

[AS68]   M. F. Atiyah and G. B. Segal. The index of elliptic operators: II. *Annals of Mathematics*, 87:531–545, 1968.

[Bor60]   Armand Borel. *Seminar on transformation groups*. Number 46 in Annals of Mathematics Studies. Princeton Univeristy Press, 1960.

[CKT]   Ana Canas da Silva, Yael Karshon, and Susan Tolman. Quantization of presymplectic manifolds and circle actions. dg-ga/9705008.

[DGMW95] H. Duistermaat, V. Guillemin, E. Meinrenken, and S. Wu. Symplectic reduction and Riemann-Roch for circle actions. *Math. Research Lett.*, 2(3):259–267, 1995.

[GK96]   Victor Guillemin and Jaap Kalkman. The Jeffrey-Kirwan localization theorem and residue operations in equivariant cohomology. *J. Reine angew. Math.*, 470:123–142, 1996.

[Kal95]   Jaap Kalkman. Cohomology rings of symplectic quotients. *J. Reine Angew. Math.*, 458:37–52, 1995.

[Kaw91]   Katsuo Kawakubo. *The Theory of Transformation Groups*. Oxford University Press, Oxford, UK, 1991.

[Kos70]   Bertram Kostant. *Modern Analysis and Applications*, volume 170 of *Lecture Notes in Mathematics*, chapter Quantization and Unitary Representations. Springer-Verlag, New York, 1970.

[Ler95]   Eugene Lerman. Symplectic cuts. *Math Research Letters*, 2:247–258, 1995.

[May96]   J. Peter May. *Equivariant Homotopy and Cohomology Theory*. Number 91 in CBMS. American Mathematical Society, 1996.

[Met]   David Metzler. A *K*-theoretic note on geometric quantization. To appear in Manuscripta Mathematica; available as preprint math.SG/9809031.

[MS95]   Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology*. Oxford University Press, New York, 1995.

[Seg68]   Graeme Segal. Equivariant K-theory. *Publ. Math. IHES*, 34:129–151, 1968.

[Sin]   Dev Sinha. Computations of complex equivariant bordism rings. Available as preprint on Hopf archive, hopf.math.purdue.edu.

[tD71]   Tammo tom Dieck. Lokalisierung äquivarianter Kohomologie-Theorien. *Math. Zeitschrift*, 121:253–262, 1971.